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A note on the uniqueness of $D = 4, N = 1$ Supergravity

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Abstract

We investigate in 4 spacetime dimensions, all the consistent deformations of the lagrangian $\mathcal{L}_2 + \mathcal{L}_{\frac{3}{2}}$, which is the sum of the Pauli-Fierz lagrangian \mathcal{L}_2 for a free massless spin 2 field and the Rarita-Schwinger lagrangian $\mathcal{L}_{\frac{3}{2}}$ for a free massless spin 3/2 field.

Using BRST cohomological techniques, we show, under the assumptions of locality, Poincaré invariance, conservation of the number of gauge symmetries and the number of derivatives on each fields, that N=1 D=4 supergravity is the only consistent interaction between a massless spin 2 and a massless spin 3/2 field. We do not assume general covariance. This follows automatically, as does supersymmetry invariance. Various cohomologies related to conservations laws are also given.

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1 Introduction

It is well appreciated that general relativity is the unique way to consistently deform the Pauli-Fierz action $\int \mathcal{L}_2$ for a free massless spin-2 field under the assumption of locality, Poincaré invariance, preservation of the number of gauge symmetries and the number of derivatives in \mathcal{L}_2 [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. This has been reconfirmed recently in [1] using BRST-cohomological techniques based on the antifield formalism, where multi-graviton theories were also included.

Supersymmetry seems to be crucial in the attempts to reconcile quantum mechanics and gravitation. It is then natural to consider the Rarita-Schwinger action $\int \mathcal{L}_{\frac{3}{2}}$ for a free massless spin 3/2 field, which describes the gravitino, the supersymmetric partner of the graviton and to subsequently analyse all the consistent deformations of the free action $I_0 = \int \mathcal{L}_2 + \mathcal{L}_{\frac{3}{2}}$.

In this paper we show that, under the assumptions of locality, Poncaré invariance, conservation of the number of gauge symmetries and the number of derivatives on each fields,

(i) the only consistent deformation of the lagrangian $\mathcal{L}_{\frac{3}{2}}$ is given to first order in the coupling constant m , by a mass term \mathcal{L}_M which is obstructed at second order (the apparition of this mass term is a well known property of the full nonlinear N=1 D=4 supergravity [12]).

(ii) the most general deformation of the lagrangian $\mathcal{L}_2 + \mathcal{L}_{\frac{3}{2}}$ to first order is given by

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_{\frac{3}{2}} + g\mathcal{L}_E + \Lambda\mathcal{L}_C + m\mathcal{L}_M + \alpha\mathcal{L}_{Int} \quad (1.1)$$

where \mathcal{L}_E is the cubic vertex of Einstein-Hilbert lagrangian (containing two derivatives : “ $\partial\partial h h h$ ”), $\mathcal{L}_C = -2h^{\mu\nu}\eta_{\mu\nu}$ is the first order deformation of the Pauli-Fierz action which corresponds to the cosmological term and \mathcal{L}_{Int} is the unique consistent deformation to first order in the coupling constant α involving the two fields (graviton and gravitino) simultaneously. This last term involves the spin connection to first order, *i.e.* converts ordinary derivatives of the (vector-)spinor field into covariant derivatives. The introduction of general covariance is not assumed and follows automatically. This strengthens previous results [13, 14, 15].

(iii) consistency to second order in the coupling constants requires relations between the constants : $g = 4\alpha$ and $3m^2 = \alpha\Lambda$ (see [16]). This leaves two independent coupling constants. As it is known that supergravity $N = 1$ $D = 4$ (with a possible cosmological term) is a consistent deformation of $\mathcal{L}_2 + \mathcal{L}_{\frac{3}{2}}$ to all orders in the coupling constants [17], we conclude that it is in fact the unique consistent interaction between a spin-2 field and a spin-3/2 field under the assumptions made above.

The use of the antifield formalism streamlines the result and systematizes the search for all possible consistent interactions of the free $I_0 = \int \mathcal{L}_2 + \mathcal{L}_{\frac{3}{2}}$ theory. We recover in a unified and esthetic way famous results on $D = 4$ $N = 1$ supergravity (relation between the coupling constants, apparition of the mass term with an abelian algebra,...)

with fewer assumptions (we do not assume general covariance, it follows automatically, as does the supersymmetric invariance). We also compute various cohomologies related to conservation laws. This will be useful for the study of different aspects of the full $D = 4$, $N = 1$ theory as well as its extensions with more supersymmetries², thanks to the importance of BRST approach for renormalization and anomalies.

1.1 Conventions

We work in 4-dimensions Minkowski space-time with the metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. $\epsilon^{0123} = -\epsilon_{0123} = 1$, $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$, the gamma matrices γ_μ are all purely imaginary. We take the matrix γ^0 to be antisymmetric and hermitian, γ^i ($i = 1, 2, 3$) are symmetric and antihermitian. The Dirac conjugate of a spinor is $\bar{\psi} = \psi^\dagger \gamma^0$ and the Majorana conjugate is $\psi^c = (\mathcal{C}\psi)^\top$. The charge conjugate matrix \mathcal{C} such that $\mathcal{C}\gamma_\mu = -\gamma_\mu^\top \mathcal{C}$ is given in our conventions by $-\gamma^0$. The Majorana spinors are such that $\bar{\psi} = \psi^c$ and are therefore real.

We define³: $\gamma^{\mu\nu} = \gamma^{[\mu}\gamma^{\nu]}$ $\gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$.

1.2 The Free Models

The Pauli-Fierz lagrangian [19] is given by :

$$\mathcal{L}_2 = -\frac{1}{2}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} + \partial_\mu h_\nu^\mu\partial_\rho h^{\rho\nu} - \partial_\nu h\partial_\rho h^{\rho\nu} + \frac{1}{2}\partial_\mu h\partial^\mu h \quad (1.2)$$

where $h_{\mu\nu}$ is a covariant symmetric tensor of rank 2. The action $S^{PF} = \int d^4x \mathcal{L}_2$ is invariant under the irreducible and abelian gauge transformations :

$$\delta_\eta h_{\mu\nu} = \partial_\mu \eta_\nu + \partial_\nu \eta_\mu \quad (1.3)$$

where η_ν is a 4-vector.

The Rarita-Schwinger lagrangian [20] is given by :

$$\mathcal{L}_{\frac{3}{2}} = -\frac{1}{2}\bar{\psi}_\alpha \gamma^{\alpha\mu\nu} \partial_\mu \psi_\nu \quad (1.4)$$

where ψ_μ is a fermionic Majorana spinor-vector. The action $S^{RS} = \int d^4x \mathcal{L}_{\frac{3}{2}}$ is invariant under the irreducible abelian gauge transformations :

$$\delta_\epsilon \psi_\mu = \partial_\mu \epsilon \quad (1.5)$$

where the gauge parameter ϵ is a fermionic Majorana spinor.

²The BRST-cohomology of the full nonlinear supergravity $D = 4$ $N = 1$ has been studied in [18]

³The notation $[a_1 \dots a_n]$ and $(a_1 \dots a_n)$ means that we consider the expression which is totally antisymmetric (resp. symmetric) in all the indices $a_1 \dots a_n$ with the normalization factor $\frac{1}{n!}$, i.e. $\gamma^{[\mu}\gamma^{\nu]} \equiv \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$

2 Cohomological reformulation

2.1 Differentials δ , γ and s

By following the general prescription of the antifield formalism [21, 22, 23], one finds that the spectrum of fields, ghosts and their associated antifields is given by :

- the field $h_{\mu\nu}$ (the graviton);
- the field ξ_ν , the ghost associated to the gauge transformations (1.3);
- the antifield $h^{*\mu\nu}$ conjugated to the field $h_{\mu\nu}$;
- the antifield $\xi^{*\nu}$ conjugated to the ghost ξ_ν .
- the field ψ_μ (the gravitino);
- the field C , the ghost associated with the gauge symmetries (1.5);
- the antifield $\bar{\psi}^{*\mu}$ conjugated to the field ψ_μ ;
- the antifield \bar{C}^* conjugated to the ghost C .

We introduce the differential γ which is the longitudinal derivative along the gauge orbits, and δ which is the Koszul-Tate differential related to the equations of motion. A grading is associated to each of these differentials : γ increases by one unit the “pure ghost number” denoted *puregh* while δ increases the “antighost number” *antigh* by one unit. The BRST-operator s is simply the sum of the two differentials :

$$s = \gamma + \delta. \quad (2.1)$$

The ghost number gh , in turn, is defined by

$$gh = \text{puregh} - \text{antigh}. \quad (2.2)$$

The action of the differentials γ and δ on all the fields of the formalism is displayed in the following array which indicates also the pureghost number, antighost number, ghost number and grassmannian parity of the various fields :

Z	$\gamma(Z)$	$\delta(Z)$	<i>puregh</i> (Z)	<i>antigh</i> (Z)	<i>gh</i> (Z)	Grassmannian parity
$h_{\mu\nu}$	$2\partial_{(\mu}\xi_{\nu)}$	0	0	0	0	0
ξ_μ	0	0	1	0	1	1
$h^{*\mu\nu}$	0	$\frac{\delta^R}{\delta h_{\mu\nu}}\mathcal{L}_2$	0	1	-1	1
$\xi^{*\nu}$	0	$-2\partial_\mu h^{*\mu\nu}$	0	2	-2	0
ψ_μ	$\partial_\mu C$	0	0	0	0	1
C	0	0	1	0	1	0
$\bar{\psi}^{*\mu}$	0	$\frac{\delta^R}{\delta \psi_\mu}\mathcal{L}_{\frac{3}{2}}$	0	1	-1	0
\bar{C}^*	0	$\partial_\mu \bar{\psi}^{*\mu}$	0	2	-2	1

It is easy to check that :

$$\begin{aligned}\gamma^2 &= \delta^2 = \gamma\delta + \delta\gamma = 0, \\ s^2 &= 0.\end{aligned}$$

2.2 Consistent deformations and cohomology

We analyse the problem of consistent deformation in the light of the master equation formalism [24]. For a review, see [22, 23, 25].

The master equation formalism associates to a local action $I_0[\phi^i]$, which is invariant under the gauge transformations ⁴

$$\delta_\epsilon \phi^i(x) = R_\alpha^i(x) \epsilon^\alpha(x) \equiv \int d^n y R_\alpha^i(x, y) \epsilon^\alpha(y), \quad (2.3)$$

a functional W depending on the original fields ϕ^i and the ghosts C^α , together with their associated antifields ϕ_i^* and C_α^* . This functional possesses the following properties :

- W is bosonic and has ghost number zero,
- W starts like $W = I_0 + \phi_i^* R_\alpha^i C^\alpha + \frac{1}{2} C_\gamma^* C_{\alpha\beta}^\gamma C^\alpha C^\beta + \frac{1}{4} \phi_i^* \phi_j^* M_{\alpha\beta}^{ij} C^\alpha C^\beta + \text{“more”}$ where “more” contains at least three ghosts,
- W fulfills the “master equation”

$$(W, W) = 0. \quad (2.4)$$

The “antibracket” (\cdot, \cdot) makes the fields ϕ^i and the ghosts C^α canonically conjugate respectively to the antifields ϕ_i^* and the antighosts C_α^* . It is defined by :

$$(A, B) = \frac{\delta^R A}{\delta \phi^i} \frac{\delta^L B}{\delta \phi_i^*} - \frac{\delta^R A}{\delta \phi_i^*} \frac{\delta^L B}{\delta \phi^i} + \frac{\delta^R A}{\delta C^\alpha} \frac{\delta^L B}{\delta C_\alpha^*} - \frac{\delta^R A}{\delta C_\alpha^*} \frac{\delta^L B}{\delta C^\alpha} \quad (2.5)$$

where the superscript R (resp. L) denotes a right (resp. left) derivative. The antibracket satisfies the graded Jacobi identity and increases the ghost number by one unit, *i.e.* $gh((A, B)) = gh(A) + gh(B) + 1$.

The master equation (2.4) is fulfilled as a consequence of the Noether identities

$$\frac{\delta I_0}{\delta \phi^i} R_\alpha^i = 0 \quad (2.6)$$

and the gauge algebra :

$$R_\alpha^j(\phi) \frac{\delta R_\beta^i(\phi)}{\delta \phi^j} - (-)^{\epsilon_\alpha \epsilon_\beta} R_\beta^j(\phi) \frac{\delta R_\alpha^i(\phi)}{\delta \phi^j} = C_{\alpha\beta}^\gamma(\phi) R_\gamma^i(\phi) + M_{\alpha\beta}^{ij}(\phi) \frac{\delta I_0}{\delta \phi^j}, \quad (2.7)$$

⁴We use the De Witt’s condensed notation : a summation over a repeated index implies also an integration over spacetime variables.

where $M_{\alpha\beta}^{ij} = -(-)^{\epsilon_{\phi^i}\epsilon_{\phi^j}} M_{\alpha\beta}^{ji}$.

Conversely, given any W solution of (2.4), one can recover the gauge-invariant action as the term independent of the ghosts in W , while the gauge transformations are defined by the terms linear in the antifields ϕ_i^* and the structure functions appearing in the gauge algebra can be read off from the terms quadratic in the ghosts. The Noether identities (2.6) are fulfilled as a consequence of the master equation, the gauge algebra (2.7) and of all the higher order identities that one can derive from them. In other words, there is complete equivalence between gauge invariance of I_0 and the existence of a solution W of the master equation. For this reason, one can reformulate the problem of consistently introducing interactions for a gauge theory as that of deforming W while maintaining the master equation (2.4).

2.3 Perturbation of the master equation

Let W_0 be the solution of the master equation for the original theory,

$$W_0 = I_0 + \phi_i^* R_\alpha^i C^\alpha, \quad (W_0, W_0) = 0. \quad (2.8)$$

Because the gauge transformations are abelian, there is no further term in W_0 . Let W be the solution of the master equation for the searched-for interacting theory, in the deformation parameter g , we have

$$W_0 = I_0 + \phi_i^* R_\alpha^i C^\alpha + O(C^2), \quad (2.9)$$

$$I = I_0 + gI_1 + g^2I_2 + \dots, \quad (2.10)$$

$$(W, W) = 0. \quad (2.11)$$

and W exists if and only if I is a consistent deformation of I_0 . Let us expand W and the master equation for W in powers of the deformation parameters g . With $W = W_0 + gW_1 + g^2W_2 + O(g^3)$, the equation $(W, W) = 0$ yields, up to order g^2 :

$$O(g^0) : \quad (W_0, W_0) = 0, \quad (2.12)$$

$$O(g^1) : \quad (W_0, W_1) = 0, \quad (2.13)$$

$$O(g^2) : \quad (W_0, W_2) = -\frac{1}{2}(W_1, W_1). \quad (2.14)$$

The first equation is fulfilled by assumption since the starting point defines a consistent theory. W_0 , the solution of the master equation for the free theory is in fact the generator of the BRST differential s :

$$sA = (W_0, A) \text{ for a functional } A. \quad (2.15)$$

the nilpotency $s^2 = 0$ follows from the master equation for W_0 and the graded Jacobi identity for the antibracket. Thus equation (2.13) expresses that W_1 is a BRST-cocycle, *i.e.* that it is “closed” under s :

$$sW_1 = 0. \quad (2.16)$$

Trivial interactions generated by field-redefinitions that reduce to the identity at order g^0 precisely correspond to cohomologically trivial solution of (2.16), i.e., correspond to “exact” A (also called “coboundaries”) of the form $A = sB$ for some B .

We thus come to the conclusion that the non-trivial consistent interactions are characterized to first order in g by the *cohomological group*⁵ $H(s)$ at ghost number zero.

Because the equation $s \int a = 0$ is equivalent to $sa + dm = 0$ (where d denotes the Cartan’s exterior differential) for some m , and $\int a = s \int b$ is equivalent to $a = sb + dn$ for some n , one denotes the corresponding cohomological group by $H^{0,n}(s|d)$ ⁶, where a, b, m, n are *local forms*, that is, differential forms with local functions as coefficients. *Local functions* depend polynomially on the fields - including the ghosts and the antifields - and their derivatives up to a finite order, in such a way that we work with functions over a finite-dimensional vectorial space, the so-called jet space.

Once a first-order deformation is given, one must investigate whether it can be extended to higher orders. It is a direct consequence of the graded Jacobi identity for the antibracket that (W_1, W_1) is BRST-closed. However, it may not be BRST-exact (in the space of local functionals). In this case, the first-order deformation W_1 is obstructed at second-order, so, it is not a good starting point. If, on the other hand, (W_1, W_1) is BRST-exact, then a solution W_2 to (2.14), which may be rewritten

$$sW_2 = -\frac{1}{2}(W_1, W_1) \quad (2.17)$$

exists. Since (W_1, W_1) has ghost number one (because the antibracket increases the ghost number by one unit), we see that obstructions to continuing a given, first order consistent interaction are measured by the cohomological group $H^{1,n}(s|d)$. Furthermore, the ambiguity in W_2 (when it exists) is a solution of the homogeneous equation $sW_2 = 0$.

The same pattern is found at higher orders : obstructions to the existence of W_k are elements of $H^{1,n}(s|d)$, while the ambiguities in W_k (when it exists) are elements of appropriate quotient spaces of $H^{0,n}(s|d)$.

To compute $H^0(s|d)$, we need the following cohomological groups : $H(\gamma)$, $H(\gamma|d)$, $H(\delta)$, $H(\delta|d)$ and $H^{inv}(\delta|d)$ as is expressed by the following theorem (see [26])

Theorem 2.1 (Cohomology of s modulo d)

$$(i) \ H^k(s|d) \simeq H_{-k}(\delta|d) \text{ with } k < 0 \quad (2.18)$$

$$(ii) \ H^k(s|d) \simeq H^k(\gamma|d, H_0(\delta)) \text{ with } k \geq 0 \quad (2.19)$$

k is the ghost number, except on the right-hand side of (2.18), where it stands for the antighost number.

⁵We recall that, given some nilpotent s , $s^2 = 0$, $H(s)$ denotes the equivalence classes of “closed” A ’s, modulo “exact” ones, i.e. the solution of $sA = 0$, modulo the equivalence relation $A' = A + sB$.

⁶More generally, we shall use the notation $H_j^{i,p}$ to denote a cohomological group for p -forms having a fixed ghost number i , and a fixed “antifield” number j . If we indicate only one superscript, it will always refer to the form degree p .

3 Cohomology of the gravitino

Following the formalism described in the previous section, we now search for all possible consistent deformations of the lagrangian $\mathcal{L}_{\frac{3}{2}}$.

3.1 $H(\gamma)$ for the gravitino.

We isolate the contractible pairs $\partial_{\mu_1 \dots \mu_p} C = \gamma(\partial_{\mu_1 \dots \mu_{p-1}} \psi_{\mu_p})$ with respect to the differential γ as in [27]. This shows that all derivatives of the ghost are γ -exact and thus are trivial in $H(\gamma)$. Furthermore, the only gauge-invariant objects constructed out of the fields ψ_μ and their derivatives are the antisymmetrized first order derivatives of the fields : $\psi_{\mu\alpha} \equiv \partial_{[\alpha} \psi_{\mu]}$ and their subsequent derivatives. Thus,

Theorem 3.1 (Cohomology of $\gamma : H(\gamma)$) *Let a be a local function⁷, $\gamma(a) = 0 \Rightarrow a = f([\psi_{\mu\alpha}], [\psi_\mu^*], [C^*], C) + \gamma b$, with some local function b .*

As the derivatives of the ghost C are γ -trivial, one can express all the invariant local functions as

$$\alpha_J([\psi_{\mu\alpha}], [\psi_\mu^*], [C^*]) \omega^J(C) \quad (3.1)$$

The α_J are local functions of $\psi_{\mu\alpha}$, ψ_μ^* , C^* and their subsequent derivatives together with forms dx^μ .

The ω^J are monomials which constitute a basis for the algebra generated by the ghost C . In opposition to the case of gravitons (see [1] section 3), this algebra is not finite-dimensional, because in our case the ghost C is bosonic.

The α_J which contain a finite number of derivatives are polynomials at each antighost numbers, because $\psi_{\mu\alpha}$ and C^* being fermionic variables, appear only a finite number of time when one fixes the number of derivatives. The occurrence of ψ_μ^* is limited by the fixation of the antighost number. In the sequel, we will only consider this case and the $\alpha_J([\psi_{\mu\alpha}], [\psi_\mu^*], [C^*])$ will be called *invariant polynomials*. In antifield number zero, the invariant polynomials are functions of the $\psi_{\mu\alpha}$ and their subsequent derivatives .

In what follows, demonstrations of theorems follow the same lines as those of similar theorems in [1] with some little adaptations.

3.2 Invariant cohomology of $d : H^{inv}(d)$

We will need to compute the cohomology of the Cartan's differential d in the space of invariant polynomials :

Theorem 3.2 (Cohomology of d in the space of invariant polynomials) *In antighost number strictly greater than 0 and in form degree less than the spacetime dimension, the cohomology of d is trivial in the space of invariant polynomials. In other words, if a is an invariant polynomial, the equation $da = 0$ (with $antigh(a) > 0$) implies a solution $a = db$ with some invariant polynomial b .*

⁷The notation $f([\phi])$ means that f is a function of ϕ and its subsequent derivatives up to a finite order.

The demonstration follows the pattern shown in [27]. Namely, split d as $d = d_1 + d_0$, where d_1 acts only on the antifields (C^* , $\psi^{*\mu}$) and d_0 acts only on the field $\psi_{\mu\nu}$. By the Poincaré Lemma (see section 4 of [23]) d_1 has no cohomology in form degree less than the space-time dimension (and in antifield number strictly greater than 0) because there is no relation among the derivatives of the antifields. By contrast, d_0 has some cohomology in the space of polynomials in $\psi_{\mu\nu}$ (see the previous section). From the triviality of the cohomology of d_1 , one easily gets $d\alpha = 0 \Rightarrow d\beta + u$, where β is an invariant polynomial, and where u is an invariant polynomial that does not involve the antifields. However, since $antigh(\alpha) > 0$, u must vanish. ■

This theorem has the following consequence :

Theorem 3.3 *If $antigh(a) > 0$ then $\gamma a + db = 0$ is equivalent (up to trivial redefinition) to $\gamma a = 0$*

The proof is similar to the demonstration in section A.1 of [1]. The difference lies in the fact that in our case, the first order derivatives of the ghosts are γ -exact. Hence we conclude easily without introducing new differential to analyse the descent equations. ■

3.3 Cohomology of δ modulo d : $H(\delta|d)$

The cohomological group $H(\delta|d)$ is related to conservations laws and more generally to the so-called characteristic cohomology through the isomorphism $H_{char}^{n-p}(d) \simeq H_p^n(\delta|d)$ [27](for a review of this aspect of the differential δ , see section 6 of [23]).

$H_l(\delta|d)$ is trivial in the space of local forms with pureghost number $l > 0$ (see [26]), we will then, in the sequel of this subsection, consider only local forms with $puregh = 0$.

Because the theory is linear and irreducible we have the (see [26])

Theorem 3.4 (Cohomology of δ modulo d : $H_p^n(\delta|d), p > 2$) *The cohomological groups $H_p^n(\delta|d)$ vanish in antifield number strictly greater than 2,*

$$H_p^n(\delta|d) = 0 \text{ with } p > 2. \quad (3.2)$$

This theorem means that all the conservation laws involving antisymmetric objects of rank strictly greater than 2 are trivial (see section 6.2 of [23]).

In antifield number two, the cohomology is given by the following theorem :

Theorem 3.5 (Cohomology of δ modulo d : $H_2^n(\delta|d)$) *A complete set of representatives of $H_2^n(\delta|d)$ is given by the antighost $C^* : \delta a_2^n + da_1^{n-1} = 0 \Rightarrow a_2^n = \lambda_r C^{*r} d^n x + \delta b_3^n + db_2^{n-1}$; where λ_r are constants.*

The proof follows similar lines as for Theorem 4.2 of [1]. Once again the demonstration is easier than in the case of graviton, thanks to the first degree in the derivatives of the spinor theory.

Let a be a solution of the cocycle condition for $H_2^n(\delta|d)$, written in dual notations, $\delta a + \partial_\mu V^\mu = 0$. Without loss of generality, one can assume that a is linear in the undifferentiated antifields (integrations by parts leaves one in the same cohomological class of $H_2^n(\delta|d)$). Thus

$$a = \lambda_r C^{*r} + b \quad (3.3)$$

where b is quadratic in the antifields $\psi^{*\mu}$ and their derivatives, and where λ_i are functions of ψ_μ and their derivatives. Because $\delta\mu \approx 0^8$, the equation $\delta a + \partial_\mu V^\mu = 0$ implies $\partial_\mu \lambda_r \psi^{*r\mu} \approx 0$. By the linear dependency of the $\psi^{*\mu}$, we conclude that $\partial_\mu \lambda_r \approx 0$. Thanks to the isomorphism $H_0^0(d|\delta)/R \simeq H_n^n(\delta|d)$ (see Theorem 6.2 of [23]) and the previous theorem $H_n^n(\delta|d) = 0$ ($n > 2$), we conclude that $\lambda_r \approx l_r$, where l_r are constants. Substituting this expression into (3.3) and noting that the term proportional to the equations of motion can be absorbed through a redefinition of b , one gets $a = l_r C^r + b'$ up to trivial terms). Now $l_r C^r$ is a solution of $\delta a + \partial_\mu V^\mu = 0$ by itself. This means that b' , which is quadratic in the $\psi^{*\mu}$ and their derivatives, must be a δ -cocycle modulo d , and hence trivial (see Theorem 11.2 of [26]). ■

3.4 Invariant cohomology of δ modulo d : $H^{inv}(\delta|d)$

In the space of invariant polynomials, we have :

Theorem 3.6 (Invariant cohomology of δ modulo d : $H^{inv}(\delta|d)$) *Let a be an invariant polynomial. If a is δ -trivial modulo d in the space of all polynomials (including non-invariant ones), $a = \delta b + dc$, then a is also δ -trivial modulo d in the space of invariant polynomials, that is to say, one can take b and c as invariant polynomials.*

The proof is similar to the one of Theorem 4.1 of [1]. It follows descent equations techniques as (A.9) of [28]. Once again, in our case we conclude easily, thanks to the first order of the spinor theory. ■

4 Consistent deformations for the gravitino

4.1 Cohomology of s modulo d

A cocycle of $H(s|d)$ is a solution of

$$sa + db = 0. \quad (4.4)$$

We expand a and b as a series indexed by the antifield number (for this section, see section A.3 of [1] and [26]) :

$$a = a_0 + a_1 + \cdots + a_k, \quad (4.5)$$

$$b = b_0 + b_1 + \cdots + b_l. \quad (4.6)$$

⁸ \approx means equal modulo equations of motion.

The two series stop at some finite antifield number, because the first-order deformation of the lagrangian is assumed to have a finite derivative order [26]. As $H_k^n(\delta|d)$ is trivial for $k \geq 3$ (see theorems 2.1 and 3.4), we can stop with $a_2 : a = a_0 + a_1 + a_2$ and $b = b_0 + b_1$ (see [26]). Using $s = \gamma + \delta$, we have the following “descent equations” :

$$\begin{aligned}\delta a_1 + \gamma a_0 + db_0 &= 0, \\ \delta a_2 + \gamma a_1 + db_1 &= 0, \\ \gamma a_2 &= 0.\end{aligned}\tag{4.7}$$

Furthermore, the term a_2 can be taken to contain only no-trivial terms of $H_2^n(\delta|d)$.

Now we have all the tools to compute $H^{0,n}(s|d)$. Note that a_0 is the deformation of the lagrangian, a_1 gives the deformation of the gauge transformations and a_2 contains information on the gauge algebra.

We start with a_2 and then “lift” it two times in order to find a_1 and a_0 .

4.2 The algebra of gauge transformations remains abelian after consistent deformation

The general solution of $\gamma a_2 = 0$ is modulo trivial terms :

$$a_2 = \alpha_J \omega^J \tag{4.8}$$

with α_J invariant polynomials. a_2 can be taken to contain non-trivial terms of $H_2^n(\delta|d)$, because if $t_2 = \alpha'_J \omega^J$ is such that $\delta t_2 + dc = 0$, then we get $st_2 + dc = 0$ (because $s = \gamma + \delta$ and $\gamma(\alpha'_J \omega^J) = 0$, hence t_2 can be taken to zero. Thus, α_J are of the type $\lambda_r C^{*r}$, where λ_r are constants. But, as ω^J contains two ghosts, a_2 must vanish because there is no Lorentz invariant expression built out of 3 spinors. We conclude that $a_2 = 0$, which means that there is no deformation of the algebra of gauge transformation : the gauge algebra remains abelian under deformation.

4.3 Deformations of the gauge transformation

Now, the equations are :

$$\begin{aligned}\delta a_1 + \gamma a_0 &= db_0, \\ \gamma a_1 &= 0.\end{aligned}\tag{4.9}$$

Equation (4.9) implies $a_1 = \alpha_J \omega^J$. As the ghost number vanish and the antighost is 1, we have to build a_1 out of the ghost C and an antifield $\psi^{*\mu}$. To respect the Lorentz invariance, one must contract the space-time indices of ψ_μ with a derivatives ∂^μ or gamma matrices. By adding a total derivative if necessary, we can put all the derivatives on the ghost C , and then the expression becomes γ -trivial. Hence, without imposing any restriction on the number of derivatives, the unique possibility is then ⁹

$$-im\bar{\psi}^{*\mu}\gamma_\mu C. \tag{4.10}$$

⁹We use $W_0 = I_0 - \bar{\psi}^{*\mu}\partial_\mu C$

This correspond to an abelian gauge transformation as anticipated by the vanishing a_2 .

$$\delta_\epsilon \psi_\mu = \mathcal{D}_\mu \epsilon = \partial_\mu \epsilon + im\gamma_\mu \epsilon. \quad (4.11)$$

Note that \mathcal{D}_μ is not a derivative (it does not respect the Leibnitz rule, this operator is known in the full nonlinear supergravity $D = 4$ $N = 1$ (see [16, 12])). m is the deformation parameter.

It is interesting to recall that, in the case of the pure spin-2 studied in [1], there was no control on the number of derivatives for the equation $\gamma a_1 = 0$. Here, the pure spin- $\frac{3}{2}$ case is much more constrained.

4.4 The only first-order consistent deformation of the lagrangian is a mass term

With $a_1 = -im\bar{\psi}_\mu^* \gamma^\mu C$, we compute

$$\delta a_1 = im\partial_\nu \bar{\psi}_\rho \gamma^{\mu\nu\rho} \gamma_\mu C. \quad (4.12)$$

As $\gamma^{\mu\nu\rho} \gamma_\mu = 2\gamma^{\nu\rho}$, the equation above gives

$$\delta a_1 = 2im\bar{C}\gamma^{\nu\rho} \partial_\nu \psi_\rho, \quad (4.13)$$

which can be written as

$$\delta a_1 = \partial_\nu (2im\bar{C}\gamma^{\nu\rho} \psi_\rho) - 2im\gamma(\bar{\psi}_\nu)\gamma^{\nu\rho} \psi_\rho. \quad (4.14)$$

Hence,

$$\delta a_1 = \partial_\nu (2im\bar{C}\gamma^{\nu\rho} \psi_\rho) - im\gamma(\bar{\psi}_\nu)\gamma^{\nu\rho} \psi_\rho \quad (4.15)$$

from which we get :

$$a_0 = im\bar{\psi}_\nu \gamma^{\nu\rho} \psi_\rho \quad (4.16)$$

$$= -\frac{i}{2}m\bar{\psi}_\mu \gamma^{\mu\nu\rho} \gamma_\nu \psi_\rho. \quad (4.17)$$

This is a mass term which is obstructed at second order [29] (this can be seen from the expression (W_1, W_1) , which is not s -exact and is hence an obstruction). This is in agreement with general belief that mass term and gauge invariance are incompatible.

The total lagrangian is

$$\mathcal{L}_{\frac{3}{2}} + \mathcal{L}_M = -\frac{1}{2}\bar{\psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \psi_\rho - \frac{i}{2}m\bar{\psi}_\mu \gamma^{\mu\nu\rho} \gamma_\nu \psi_\rho \quad (4.18)$$

$$= -\frac{1}{2}\bar{\psi}_\mu \gamma^{\mu\nu\rho} \mathcal{D}_\nu \psi_\rho \quad (4.19)$$

and is invariant under the gauge transformation $\delta_\epsilon \psi_\mu = \mathcal{D}_\mu \epsilon$ to first order in the coupling constant m . The gravitino-mass term was already known in the full nonlinear theory of $N=1$ $D=4$ supergravity [12, 16].

5 Consistent interactions between a spin-2 and a spin-3/2 field

In this section, we will search all the consistent deformations of $\mathcal{L}_2 + \mathcal{L}_{\frac{3}{2}}$ to first order in the deformation parameters under the assumptions made in the introduction, namely locality, Poincaré invariance, conservation of the number of gauge symmetries and of the number of derivatives acting on the fields.

As we found in the previous section, the only deformations \mathcal{L}_M involve only the gravitino ψ_μ . The deformations involving only the graviton were computed in [1]¹⁰. All we have to do is to find interactions terms involving both the gravitino ψ_μ and the graviton $h_{\mu\nu}$ simultaneously.

5.1 Cohomology of $\mathcal{L}_2 + \mathcal{L}_{\frac{3}{2}}$

As the theory is the sum of two free lagrangians, we have

$$H(\gamma) = H(\gamma_2) \otimes H(\gamma_{\frac{3}{2}}). \quad (5.1)$$

Using result of our section 3 and of [1] on the cohomology of γ , if a is a local function of the fields (including all the spectrum of the Batalin-Vilkovisky formalism) , then¹¹ :

$$a = \alpha_{JJ'}([K], [h^*], [\xi^*], [\partial_{[\mu}\psi_{\nu]}], [\psi_*], [C_*])\omega^{J'}(C)\omega^J(\xi_\mu, \partial_{[\mu}\xi_{\nu]}). \quad (5.2)$$

The following theorems are quite direct by using the similar theorem for the two free lagrangians :

Theorem 5.1 (Invariant cohomology of $d : H^{inv}(d)$) *In form degree less than the spacetime dimension and in antifield number strictly greater than zero, the cohomology of d in the space of invariant polynomials is trivial.*

Theorem 5.2 (Cohomology of δ modulo $d : H_p^n(\delta|d) = 0$ for $p > 2$.) *$H_p^n(\delta|d)$ is trivial in antifield number strictly greater than 2,*

$$H_p^n(\delta|d) = 0 \quad \text{for } p > 2. \quad (5.3)$$

Theorem 5.3 (Cohomology of δ modulo $d : H_2^n(\delta|d)$) *The cohomology of $H_2^n(\delta|d)$ is generated by the antighosts C^{*r}, ξ_μ^* :*

$$\delta a_2^n + \delta a_1^{n-1} = 0 \Rightarrow a_2^n = (\lambda_r C^{*r} + f_\mu \xi^{*\mu}) dx^0 dx^1 \cdots dx^{n-1} + \delta b_3^n + db_2^{n-1} \quad (5.4)$$

where λ_r and f_μ are constants.

Theorem 5.4 (Invariant cohomology of δ modulo $d : H^{inv}(\delta|d)$) *Let a be δ -trivial modulo d and γ -invariant, $a = \delta b + dc$, then a is δ -trivial in the space of invariant polynomials. That is to say, one can choose b and c in the space of invariant polynomials.*

¹⁰This was $\mathcal{L}_E + \mathcal{L}_C$, where \mathcal{L}_C is the cosmological term with no derivative of the field and \mathcal{L}_E is the term of the Einstein-Hilbert action of third order in the field $h_{\mu\nu}$, this term has two derivatives.

¹¹ K stands for the linearized Riemann tensor

Cohomology of s modulo d

Let a be ghost number zero solution of

$$sa + db = 0. \quad (5.5)$$

For the same reason as in the case of the gravitino, we can expand a and b according to the antighost number as :

$$a = a_0 + a_1 + a_2, \quad (5.6)$$

$$b = b_0 + b_1, \quad (5.7)$$

with $s = \gamma + \delta$ we get the descent equations :

$$\delta a_1 + \gamma a_0 = db_0, \quad (5.8)$$

$$\delta a_2 + \gamma a_1 = db_1, \quad (5.9)$$

$$\gamma a_2 = 0. \quad (5.10)$$

Once more, we recall that a_0 is the deformation of the lagrangian, a_1 gives the deformation of the gauge transformations and a_2 contains information on the gauge algebra.

5.2 Deformations of the algebra of gauge symmetries

The general solution of $\gamma a_2 = 0$ which involve mixed terms (that is with spin-2 and spin- $\frac{3}{2}$) and respect Poincaré invariance is, modulo trivial terms,¹² :

$$a_2 = \alpha \frac{1}{4} \xi^{*\alpha} \bar{C} \gamma_\alpha C + \frac{\beta}{4} \partial_{[\alpha} \xi_{\beta]} \bar{C}^* \gamma^{\alpha\beta} C + \frac{\lambda}{4} \xi_\alpha \bar{C}^* \gamma^\alpha C, \quad (5.11)$$

where the factors in front of the coefficients are chosen for further convenience.

5.3 Deformation of the gauge symmetries

In this subsection we study whether one can lift a_2 to a certain a_1 by looking at solution of $\delta a_2 + \gamma a_1 = db_1$.

5.3.1 Lift of $a_2 = \frac{1}{4} \xi_\alpha^* \bar{C} \gamma^\alpha C$ to a_1

Let $a_2 = \frac{1}{4} \xi_\alpha^* \bar{C} \gamma^\alpha C$, then

$$\delta a_2 = -\frac{1}{2} \partial_\beta h^{*\alpha\beta} \bar{C} \gamma_\alpha C \quad (5.12)$$

$$= \partial_\beta \left(-\frac{1}{2} h^{*\alpha\beta} \bar{C} \gamma_\alpha C \right) + h^{*\alpha\beta} (\partial_\beta \bar{C}) \gamma_\alpha C \quad (5.13)$$

$$= \partial_\beta \left(-\frac{1}{2} h^{*\alpha\beta} \bar{C} \gamma_\alpha C \right) - \gamma (h^{*\alpha\beta} \bar{\psi}_\beta \gamma_\alpha C) \quad (5.14)$$

from which we get

$$a_1 = h^{*\alpha\beta} \bar{\psi}_\alpha \gamma_\beta C. \quad (5.15)$$

modulo γ -exact terms .

¹²As in the case of gravitino we can ignore trivial terms of $H_2^n(\delta|d)$

5.3.2 Lift of $a_2 = \partial_{[\alpha}\xi_{\beta]}\bar{C}_*\gamma^{\alpha\beta}C$ to a_1

Let $a_2 = \partial_{[\alpha}\xi_{\beta]}\bar{C}_*\gamma^{\alpha\beta}C$, as the matrix $\gamma^{\alpha\beta}$ is antisymmetric with respect to $\alpha\beta$ we write $a_2 = \partial_{\alpha}\xi_{\beta}\bar{C}_*\gamma^{\alpha\beta}C$.

$$\delta a_2 = \delta(\partial_{\alpha}\xi_{\beta}\bar{C}_*\gamma^{\alpha\beta}C) \quad (5.16)$$

$$= \partial_{\alpha}\xi_{\beta}\partial_{\lambda}\bar{\psi}_*^{\lambda}\gamma^{\alpha\beta}C \quad (5.17)$$

$$= \partial_{\lambda}(\partial_{\alpha}\xi_{\beta}\bar{\psi}_*^{\lambda}\gamma^{\alpha\beta}C) - \partial_{\lambda\alpha}\xi_{\beta}\bar{\psi}_*^{\lambda}\gamma^{\alpha\beta}C - \partial_{\alpha}\xi_{\beta}\bar{\psi}_*^{\lambda}\gamma^{\alpha\beta}\partial_{\lambda}C. \quad (5.18)$$

As

$$\partial_{\lambda\alpha}\xi_{\beta}\bar{\psi}_*^{\lambda}\gamma^{\alpha\beta}C = \gamma\left[\frac{1}{2}(\partial_{\lambda}h_{\alpha\beta} + \partial_{\alpha}h_{\beta\lambda} - \partial_{\beta}h_{\lambda\alpha})\bar{\psi}_*^{\lambda}\gamma^{\alpha\beta}C\right] \quad (5.19)$$

$$= \gamma(\partial_{\alpha}h_{\lambda\beta}\bar{\psi}_*^{\lambda}\gamma^{\alpha\beta}C) \quad (5.20)$$

and

$$-\partial_{\alpha}\xi_{\beta}\bar{\psi}_*^{\lambda}\gamma^{\alpha\beta}\partial_{\lambda}C = \gamma(\partial_{\alpha}\xi_{\beta}\bar{\psi}_*^{\lambda}\gamma^{\alpha\beta}\psi_{\lambda}) \quad (5.21)$$

we obtain :

$$a_1 = \partial_{\alpha}h_{\lambda\beta}\bar{\psi}_*^{\lambda}\gamma^{\alpha\beta}C - \partial_{\alpha}\xi_{\beta}\bar{\psi}_*^{\lambda}\gamma^{\alpha\beta}\psi_{\lambda} \quad (5.22)$$

$$= \partial_{\alpha}h_{\lambda\beta}\bar{C}\gamma^{\alpha\beta}\psi_{\lambda}^* - \partial_{\alpha}\xi_{\beta}\bar{\psi}_{\lambda}\gamma^{\alpha\beta}\psi_{\lambda}^*. \quad (5.23)$$

5.3.3 Obstruction for $a_2 = \xi_{\alpha}\bar{C}^*\gamma^{\alpha}C$

Let $a_2 = \xi_{\alpha}\bar{C}_*\gamma^{\alpha}C$,

$$\delta a_2 = -\xi_{\alpha}\partial_{\beta}\bar{\psi}_*^{\beta}\gamma^{\alpha}C \quad (5.24)$$

$$= \partial_{\beta}(-\xi_{\alpha}\bar{\psi}_*^{\beta}\gamma^{\alpha}C) + \partial_{\beta}\xi_{\alpha}\bar{\psi}_*^{\beta}\gamma^{\alpha}C + \xi_{\alpha}\bar{\psi}_*^{\beta}\gamma^{\alpha}\partial_{\beta}C \quad (5.25)$$

$$= \partial_{\beta}(-\xi_{\alpha}\bar{\psi}_*^{\beta}\gamma^{\alpha}C) + \partial_{[\beta}\xi_{\alpha]}\bar{\psi}_*^{\beta}\gamma^{\alpha}C + \partial_{(\beta}\xi_{\alpha)}\bar{\psi}_*^{\beta}\gamma^{\alpha}C + \xi_{\alpha}\bar{\psi}_*^{\beta}\gamma^{\alpha}\partial_{\beta}C. \quad (5.26)$$

The last two terms are γ -exact :

$$\xi_{\alpha}\bar{\psi}_*^{\beta}\gamma^{\alpha}\partial_{\beta}C = -\gamma(\xi_{\alpha}\bar{\psi}_*^{\beta}\gamma^{\alpha}\psi_{\beta}) \quad (5.27)$$

$$\partial_{(\beta}\xi_{\alpha)}\bar{\psi}_*^{\beta}\gamma^{\alpha}C = \gamma(h_{\beta\alpha}\bar{\psi}_*^{\beta}\gamma^{\alpha}C). \quad (5.28)$$

The term $\partial_{[\beta}\xi_{\alpha]}\bar{\psi}_*^{\beta}\gamma^{\alpha}C$ is clearly non-trivial in $H(\gamma|d)$. That can be seen as follows : Suppose that

$$\partial_{[\beta}\xi_{\alpha]}\bar{\psi}_*^{\beta}\gamma^{\alpha}C = \gamma u + \partial_{\mu}v^{\mu} \quad (5.29)$$

by taking the Euler-Lagrange derivative with respect to $\bar{\psi}_*^{\mu}$ of both side of the previous relation, one gets that $\frac{\delta^L}{\delta\bar{\psi}_*^{\mu}}\partial_{[\beta}\xi_{\alpha]}\bar{\psi}_*^{\beta}\gamma^{\alpha}C$ is γ -exact, because γ commutes with $\frac{\delta}{\delta\bar{\psi}_*^{\mu}}$ and $\frac{\delta}{\delta\bar{\psi}_*^{\mu}}\partial_{\mu}v^{\mu} = 0$.

In fact, one gets : $\frac{\delta}{\delta\bar{\psi}_*^{\mu}}\partial_{[\beta}\xi_{\alpha]}\bar{\psi}_*^{\beta}\gamma^{\alpha}C = -\partial_{[\mu}\xi_{\alpha]}\gamma^{\alpha}C$ which clearly not γ -exact as term of the form $\partial_{[\mu}\xi_{\alpha]}$ are not γ_2 -exact (see section 3 of [1]).

The most general mixed term for a_1 is thus,

$$a_1 = \alpha h^{*\alpha\beta} \bar{\psi}_\beta \gamma_\alpha C + \frac{\beta}{4} (\partial_\alpha h_{\lambda\beta} \bar{C} \gamma^{\alpha\beta} \psi_*^\lambda - \partial_\alpha \xi_\beta \bar{\psi}_\lambda \gamma^{\alpha\beta} \psi_*^\lambda) + \gamma e. \quad (5.30)$$

Note that lifting a_2 to a_1 has reduced the number of free parameters for the interaction part of W from 3 (α, β, λ) to 2 (α, β) thanks to the obstruction of $a_2 = \xi_\alpha \bar{C}^* \gamma^\alpha C$.

5.4 Deformation of the lagrangian

$\mathcal{L}_{Int} = a_0$ is the solution of $\delta a_1 + \gamma a_0 = db_0$.

Starting with

$$\begin{aligned} a_1 &= \alpha (h^{*\alpha\beta} \bar{\psi}_\beta \gamma_\alpha C) + \\ &+ \frac{\beta}{4} (\partial_\alpha h_{\lambda\beta} \bar{\psi}^{*\lambda} \gamma^{\alpha\beta} C - \partial_\alpha \xi_\beta \bar{\psi}^{*\lambda} \gamma^{\alpha\beta} \psi_\lambda) \end{aligned} \quad (5.31)$$

we get

$$\begin{aligned} \delta a_1 &= \alpha (\square h^{\alpha\beta} + \partial^{\alpha\beta} h - \partial^\alpha \partial_\rho h^{\rho\beta} - \partial^\beta \partial_\rho h^{\rho\alpha} + \\ &+ \eta^{\alpha\beta} \partial_{\rho\lambda} h^{\rho\lambda} - \eta^{\alpha\beta} \square h) \bar{\psi}_\beta \gamma_\alpha C + \\ &+ -\frac{\beta}{4} \underbrace{(\partial_\alpha h_{\lambda\beta} \partial_\mu \bar{\psi}_\nu \gamma^{\mu\nu\lambda} \gamma^{\alpha\beta} C + \partial_\alpha \xi_\beta \partial_\mu \bar{\psi}_\nu \gamma^{\mu\nu\lambda} \gamma^{\alpha\beta} \psi_\lambda)}_I. \end{aligned} \quad (5.32)$$

Up to total derivatives, the expression I writes

$$I = -[\partial_{\mu\alpha} h_{\beta\lambda} \bar{\psi}_\nu \gamma^{\mu\nu\lambda} \gamma^{\alpha\beta} C + \partial_\alpha h_{\beta\lambda} \bar{\psi}_\nu \gamma^{\mu\nu\lambda} \gamma^{\alpha\beta} (\gamma^\mu \psi_\mu)]. \quad (5.33)$$

Using

$$\begin{aligned} \gamma^{\mu\nu\lambda} \gamma^{\alpha\beta} &= [\gamma^{\mu\nu\lambda} \gamma^{\alpha\beta}]^{(3)} + [\gamma^{\mu\nu\lambda} \gamma^{\alpha\beta}]^{(1)} \\ &= [\eta^{\alpha\lambda} \gamma^{\beta\mu\nu} + \eta^{\alpha\mu} \gamma^{\beta\nu\lambda} + \eta^{\alpha\nu} \gamma^{\beta\lambda\mu} + \\ &+ (-)\eta^{\beta\lambda} \gamma^{\alpha\mu\nu} - \eta^{\beta\mu} \gamma^{\alpha\nu\lambda} - \eta^{\beta\nu} \gamma^{\alpha\lambda\mu}] + \\ &+ [\eta^{\beta\lambda} \eta^{\alpha\mu} \gamma^\nu + \eta^{\beta\mu} \eta^{\alpha\nu} \gamma^\lambda + \eta^{\beta\nu} \eta^{\alpha\lambda} \gamma^\mu \\ &- \eta^{\beta\lambda} \eta^{\alpha\nu} \gamma^\mu - \eta^{\beta\nu} \eta^{\alpha\mu} \gamma^\lambda - \eta^{\beta\mu} \eta^{\alpha\lambda} \gamma^\nu] \end{aligned} \quad (5.34)$$

we have also

$$\begin{aligned} I &= \partial_\mu V^\mu + [(\square h^{\alpha\beta} + \partial^{\alpha\beta} h - \partial^\alpha \partial_\rho h^{\rho\beta} - \partial^\beta \partial_\rho h^{\rho\alpha} + \\ &+ \eta^{\alpha\beta} \partial_{\rho\lambda} h^{\rho\lambda} - \eta^{\alpha\beta} \square h) \bar{\psi}_\beta \gamma_\alpha C] - \partial_{\mu\alpha} h_{\beta\lambda} \bar{\psi}_\nu [\gamma^{\mu\nu\lambda} \gamma^{\alpha\beta}]^{(3)} C + \\ &+ \frac{1}{2} \gamma (\partial_\alpha h_{\beta\lambda} \bar{\psi}_\nu [\gamma^{\mu\nu\lambda} \gamma^{\alpha\beta}]^{(1)} \psi_\mu) + \\ &- \partial_\alpha \xi_\beta \partial_\mu \bar{\psi}_\nu [\gamma^{\mu\nu\lambda} \gamma^{\alpha\beta}]^{(1)} \psi_\lambda - \partial_{\mu\alpha} h_{\beta\lambda} \bar{\psi}_\nu [\gamma^{\mu\nu\lambda} \gamma^{\alpha\beta}]^{(3)} \partial_\mu C. \end{aligned} \quad (5.35)$$

Also, as $\bar{\psi}_\nu[\gamma^{\mu\nu\lambda}\gamma^{\alpha\beta}]\psi_\mu = 0$, we obtain the following result for δa_1 :

$$\begin{aligned}\delta a_1 &= \partial_\mu V^\mu + \gamma \left[\frac{\beta}{8} \bar{\psi}_\mu \gamma^{\mu\nu\lambda} \overset{(1)}{\omega}_\nu \psi_\lambda \right] + \\ &+ \left(\alpha - \frac{\beta}{4} \right) [(\square h^{\alpha\beta} + \partial^{\alpha\beta} h - \partial^\alpha \partial_\rho h^{\rho\beta} - \partial^\beta \partial_\rho h^{\rho\alpha} + \\ &+ \eta^{\alpha\beta} \partial_{\rho\lambda} h^{\rho\lambda} - \eta^{\alpha\beta} \square h) \bar{\psi}_\beta \gamma_\alpha C] + \\ &+ -\frac{\beta}{4} \partial_\alpha h_{\beta\lambda} \partial_\mu \bar{\psi}_\nu [\gamma^{\mu\nu\lambda} \gamma^{\alpha\beta}] C - \frac{\beta}{4} \partial_\alpha \xi_\beta \partial_\mu \bar{\psi}_\nu [\gamma^{\mu\nu\lambda} \gamma^{\alpha\beta}] \psi_\lambda, \end{aligned} \quad (5.36)$$

where $\overset{(1)}{\omega}_\lambda = \partial_\alpha h_{\beta\lambda} \gamma^{\beta\alpha}$ is the spin connection at first order in the field $h_{\mu\nu}$ ¹³. The previous expression has the advantage to give us already a part of a_0 , the deformation of the lagrangian :

$$a_0 = \dots - \frac{\beta}{8} \bar{\psi}_\mu \gamma^{\mu\nu\lambda} \overset{(1)}{\omega}_\nu \psi_\lambda. \quad (5.37)$$

We may also get rid of the terms beginning with $\square h^{\alpha\beta} \bar{\psi}_\beta \gamma_\alpha C$ by choosing $\alpha = \frac{\beta}{4}$. Once those simplifications are done and the first piece of a_0 is discarded, we are left with two terms in δa_1 , that we have to express as γ -exact terms plus total derivatives. These two terms contain three gamma matrices explicetely. In order to express them as γ -exact terms modulo total derivatives, we give a basis of a_0 -terms which correspond to the two remaining pieces of δa_1 . This basis reads

$$\begin{aligned}\{Q_\Delta\}_{0 \leq \Delta \leq 3} &= \{ h \bar{\psi}_\alpha \gamma^{\alpha\mu\nu} \partial_\mu \psi_\nu, h_{\alpha\beta} \bar{\psi}_\mu \gamma^{\alpha\mu\nu} \partial^\beta \psi_\nu, \\ &h_{\alpha\beta} \bar{\psi}_\mu \gamma^{\alpha\mu\nu} \partial_\nu \psi^\beta, h_{\alpha\beta} \bar{\psi}^\beta \gamma^{\alpha\mu\nu} \partial_\mu \psi_\nu \}. \end{aligned} \quad (5.38)$$

We then compute $\gamma(\alpha^\Delta Q_\Delta)$, and try to match this with $\delta a_1^{remaining} + \partial_\mu V^\mu$. This gives a system of equations for the coefficients α^Δ which is solved for the following values, following the same order as for the Q_Δ :

$$\alpha^\Delta = \left\{ -\frac{\beta}{4}, -\frac{\beta}{4}, \frac{\beta}{4}, \frac{\beta}{4} \right\}. \quad (5.39)$$

Actually the system is not completely solved, namely there remains a term in $-\frac{1}{4} \gamma(\beta Q_0 + \beta Q_1 - \beta Q_2 - \beta Q_3)$ which does not match anything in the remaining terms of δa_1 . The complete equation $\delta a_1 + \gamma a_0$ is a total derivative modulo the following term :

$$-2\beta \xi^\beta \partial_{[\alpha} \bar{\psi}_{\beta]} \gamma^{\alpha\mu\nu} \partial_{[\mu} \psi_{\nu]}. \quad (5.40)$$

It obviously belongs to $H(\gamma)$. However, a rapid check immediately tells us that this term can be absorbed in a_1 , because is δ -exact. The new part that a_1 acquires is

$$a_1 \rightarrow a_1 + 2\beta \xi^\lambda \bar{\psi}^{*\alpha} \partial_{[\alpha} \psi_{\lambda]}. \quad (5.41)$$

¹³Note the slight abuse of notation : by ω_μ we really mean $\omega_\mu^{ab} \gamma_{ab}$, with ω_μ^{ab} the spin connection.

It is amusing to see what this term corresponds to. The equation $\delta a_2 + \gamma a_1 = dc_1$ gave us the a_1 in (5.30), modulo a solution \bar{a} of the homogeneous equation $\gamma \bar{a}_1 + dc_1 = 0$. This last equation, being of strictly positive antighost number, is equivalent to $\gamma \bar{a}_1 = 0$, as theorem 3.3 learns us. Now, asking that the deformed lagrangian does not bring more than one derivative on the ψ_μ , the only candidate a_1 belonging to $H(\gamma)$ is precisely $\xi^\beta \bar{\psi}^{*\alpha} \partial_{[\alpha} \psi_{\lambda]}$, if we don't use any gamma matrices, and demanding Lorentz invariance. The final a_0 is finally

$$\begin{aligned} a_0 = & -\frac{\beta}{2} \left[\bar{\psi}_\alpha \gamma^{\alpha\mu\nu} \frac{1}{4} \omega_\mu^{(1)} \psi_\nu + \right. \\ & + (-) \bar{\psi}_\alpha \gamma^{\alpha\mu\nu} \frac{h^\beta_\mu}{2} \partial_\beta \psi_\nu - \bar{\psi}_\alpha \gamma^{\alpha\mu\nu} \frac{h^\beta_\nu}{2} \partial_\mu \psi_\beta - \bar{\psi}_\beta \frac{h^\beta_\alpha}{2} \gamma^{\alpha\mu\nu} \partial_\mu \psi_\nu + \\ & \left. + \frac{h}{2} \bar{\psi}_\alpha \gamma^{\alpha\mu\nu} \partial_\mu \psi_\nu \right]. \end{aligned} \quad (5.42)$$

This is indeed the right cubic vertex of $N = 1, D = 4$ supergravity. To convince the reader we recall that the complete lagrangian is

$$\mathcal{L} \propto e \bar{\psi}_\alpha \gamma^{abc} e_a^\alpha e_b^\mu e_c^\nu D_\mu \psi_\nu, \quad (5.43)$$

while at first order

$$\begin{aligned} g_{\mu\nu} &\equiv e_\mu^a e_{a\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \\ e_\mu^a &= \delta_\mu^a + \frac{1}{2} h_\mu^a, \\ e_a^\mu &= \delta_a^\mu - \frac{1}{2} h_a^\mu, \\ e &\equiv \sqrt{-g} = 1 + \frac{1}{2} h_{\mu\nu} \eta^{\mu\nu} \equiv 1 + \frac{1}{2} h. \end{aligned} \quad (5.44)$$

The complete lagrangian contains the covariant derivative

$$D_\mu \psi_\nu = \partial_\mu \psi_\nu + \frac{1}{4} \omega_\mu \psi_\nu. \quad (5.45)$$

Taking into account the results for the gravitino alone, the non-trivial expression of a_1 is :

$$a_1 = -\frac{\beta}{4} h^{*\alpha\beta} \bar{\psi}_\beta \gamma_\alpha C + \frac{\beta}{4} (\partial_\alpha h_{\lambda\beta} \bar{C} \gamma^{\alpha\beta} \psi_*^\lambda - \partial_\alpha \xi_\beta \bar{\psi}_\lambda \gamma^{\alpha\beta} \psi_*^\lambda) - im \bar{\psi}_\mu^* \gamma^\mu C + 2\beta \bar{\psi}^{*\alpha} \xi^\lambda \partial_{[\alpha} \psi_{\lambda]}. \quad (5.46)$$

The last term $+2\beta \bar{\psi}^{*\alpha} \xi^\lambda \partial_{[\alpha} \psi_{\lambda]}$ gives us the Lie derivative of ψ_α along the vector ξ^λ : after partial integration of $\beta \bar{\psi}^{*\alpha} \xi^\lambda \partial_\alpha \psi_\lambda$ we get $-\beta \partial_\alpha \bar{\psi}^{*\alpha} \xi^\lambda \psi_\lambda$ which is δ -exact and is thus absorbed through a trivial redefinition $a_2 \rightarrow a_2 + \gamma[\beta \bar{C}^* \xi^\lambda \psi_\lambda]$, plus $-\beta \bar{\psi}^{*\alpha} \partial_\alpha \xi^\lambda \psi_\lambda$ which, combined with $-\beta \bar{\psi}^{*\alpha} \xi^\lambda \partial_\lambda \psi_\alpha$ is the Lie derivative of the covector ψ_α .

The existence of an interaction vertex implies automatically the Lie derivative of the gravitino as a gauge symmetry of the theory.

6 Consistency to second order and uniqueness of the deformation

Putting the results of the two previous sections and of [1] together, we find that the most general consistent deformation of the lagrangian $\mathcal{L}_2 + \mathcal{L}_{\frac{3}{2}}$, which is local, respect Poincaré invariance and the number of gauge transformations and derivative of each field is :

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_{\frac{3}{2}} + g\mathcal{L}_E + \alpha\mathcal{L}_{int} + m\mathcal{L}_M + \Lambda\mathcal{L}_C. \quad (6.1)$$

The master equation at second order in the couplings constants is : $(W_1, W_1) = -2sW_2$. This implies that (W_1, W_1) must be BRST-trivial. By looking at the term of maximal antighost number $(g\xi^{*\alpha}\xi^\beta\partial_{[\beta}\xi_{\alpha]} + \alpha(\xi^{*\alpha}\bar{C}\gamma_\alpha C + \partial^\alpha\xi^\beta\bar{C}^{*\gamma}\gamma_{\alpha\beta}C)$ this gives the relations

$$4\alpha^2 - g\alpha = 0. \quad (6.2)$$

The solution $\alpha = 0$ is not consider because it forbids interactions, we then have

$$g = 4\alpha. \quad (6.3)$$

On the other hand, if we want (W_1, W_1) to be BRST-trivial, we must have an other relation :

$$\alpha\Lambda - 3m^2 = 0. \quad (6.4)$$

Hence, consistency to second order leaves us with only two free coupling constants : g and m .

6.1 Analysis of the deformation

We will now show that the deformed theory corresponds to $D = 4$ $N = 1$ supergravity.

6.1.1 Analysis of the lagrangian

First note that the deformed lagrangian is now :

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_{\frac{3}{2}} + g\mathcal{L}_E + 4g\mathcal{L}_{int} + m\mathcal{L}_M + \frac{12m^2}{g}\mathcal{L}_C. \quad (6.5)$$

This corresponds to the linearized lagrangian of $D = 4$ $N = 1$ supergravity with a cosmological constant $\Lambda = \frac{3m^2}{\alpha} = \frac{12m^2}{g}$.

6.1.2 Analysis of the gauge symmetries

As a_1 is related to the gauge transformations of the fields. The most general non-trivial a_1 is :

$$a_1 = -\frac{g}{4}h^{*\alpha\beta}\bar{\psi}_\beta\gamma_\alpha C + \frac{g}{4}(\partial_\alpha h_{\lambda\beta}\bar{C}\gamma^{\alpha\beta}\psi_*^\lambda - \partial_\alpha\xi_\beta\bar{\psi}_\lambda\gamma^{\alpha\beta}\psi_*^\lambda) + 2g\xi^\lambda\bar{\psi}^{*\alpha}\partial_{[\alpha}\psi_{\lambda]} - im\bar{\psi}^{*\mu}\gamma_\mu C. \quad (6.6)$$

This corresponds to the gauge transformations for the gravitino :

$$\delta_{\epsilon,\zeta}\psi_\lambda = \partial_\lambda\epsilon + \frac{g}{4}\partial_\alpha h_{\beta\lambda}\gamma^{\beta\alpha}C + \frac{g}{4}\partial_\alpha\zeta_\beta\gamma^{\alpha\beta}\psi_\lambda + g(\zeta^\alpha\partial_\alpha\psi_\lambda + \partial_\lambda\zeta^\alpha\psi_\alpha) + im\gamma_\lambda\epsilon \quad (6.7)$$

where ϵ is a spinor and η a 4-vector. The first two terms on the right-hand side correspond to the linearized covariant derivatives :

$$D_\lambda = \partial_\lambda + \frac{1}{4}\overset{(1)}{\omega}_\lambda \quad (6.8)$$

where $\overset{(1)}{\omega}_\lambda = -\partial_\mu h_{\lambda\beta}\gamma^{\mu\beta}$. The third term is a linearized Lorentz transformation. The fourth term is the Lie derivative of the covector ψ_α along the diffeomorphism vector. The last one is the mass term. For the graviton, we get :

$$\delta_\epsilon h_{\mu\nu} = -\frac{g}{2}(\bar{\psi}_\mu\gamma_\nu\epsilon + \bar{\psi}_\nu\gamma_\mu\epsilon) \quad (6.9)$$

which is the linearized supergravity gauge symmetry for the graviton.

6.1.3 Analysis of the algebra of gauge symmetries

Commutators of gauge transformations are related to a_2 .

With

$$a_2 = \frac{g}{4}\left(\frac{1}{2}\xi^{*\alpha}\bar{C}\gamma_\alpha C - \partial_{[\alpha}\xi_{\beta]}\bar{C}^*\gamma^{\alpha\beta}C\right) \quad (6.10)$$

we get :

$$[\epsilon_1, \epsilon_2]\eta_\alpha = \frac{1}{2}\bar{\epsilon}_1\gamma_\alpha\epsilon_2 \quad (6.11)$$

and

$$[\eta_\alpha, \epsilon]\epsilon' = -\partial_{[\alpha}\eta_{\beta]}\gamma^{\alpha\beta}\epsilon' \quad (6.12)$$

This is indeed the $N = 1$ $D = 4$ supersymmetric algebra.

Hence we have proved that our deformation corresponds to $D = 4$ $N = 1$ linearized supergravity with a possible cosmological term . As we know that this latter is consistent to all orders [17], we have proved that supersymmetry (through supergravity) is the only way to introduce consistent interactions between a massless spin 2 and a massless spin 3/2-field under the assumptions stated in the introduction.

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